Self-interaction and slow dynamics

Toyonori Munakata* and Masayuki Uranagase†
Department of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University, Kyoto 606-8501, Japan
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A simple spin system with self-interaction is proposed and studied where the interaction among spins is taken into account within a mean field model. The self-interaction, which turns out to be a kind of activation energy, gives rise to slow relaxation due to disorder introduced to the self-interaction. In a thermodynamic limit ($N\to\infty$), a phase diagram is obtained and a spin-spin time correlation function (TCF) is calculated exactly to show that relaxation time diverges at low temperature. Dynamical property of a finite system, such as relaxation of the TCF, is studied by numerically solving a master equation.

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I. INTRODUCTION

With use of statistical mechanical methods, such as the canonical ensemble theory, much progress has been achieved in the field of phase transition [1,2]. To discuss dynamical properties such as relaxation to equilibrium states or time correlation functions, related to fluctuations around equilibrium states, Glauber dynamics [3] has often been employed if we take, e.g., spin systems which we study below.

In this Brief Report we consider a spin system with self-interaction (SI). So long as we know, SI has not been paid much attention. For example, SI, which is represented by $I_i s_i^n$ in Eq. (1) below, was carefully omitted in the Hopfield model of neural network, so that an energy function H and statistical mechanical method based on H can be usefully applied [4,5]. We study the system with SI using a standard heat-bath algorithm [6,7] and give both static (a phase diagram) and dynamic results (spin-spin time correlation function, TCF hereafter). First we consider the system in the thermodynamic limit and then turn to effects of finite system size.

In Sec. II we introduce our model and its dynamics. It is noted that *disorder* is introduced to the SI strength, I_i, by sampling it according to the probability distribution, $p(I_i)$, and that this causes slow relaxation.

Section III is devoted to studying a phase diagram within the mean field treatment. In Sec. IV a spin-spin TCF is discussed, where effects of finite system size are also considered. In Sec. V some discussions are given to relate our model with other systems recently studied in econophysics [6] and information processing [8].

II. MODEL

We consider a system consisting of N Ising spins $\{s_i=\pm 1\}$ $(i=1,2,\ldots,N)$, each s_i feeling at time n the field

$$f_i^n=\sum_{j(\neq i)} J_{i,j}s_j^n+I_i s_i^n \equiv h_i^n+I_i s_i^n, \tag{1}$$

where $J_{i,j}$ and I_i denote the interaction strength of the pair (i,j) and the self-interaction (SI), respectively.

*munakata@amp.i.kyoto-u.ac.jp
†masayuki@amp.i.kyoto-u.ac.jp

Once the field f_i^n is given, dynamics of the spin s_i is governed by the transition probability $p(s_i^{n+1}=\pm 1|s_i^n=1)$ to observe $s_i^{n+1}=\pm 1$ starting from $s_i^n=1$ of the form [7]

$$p(s_i^{n+1}=\pm 1|s_i^n=1)=1/\{1+\exp(\mp 2[h_i^n+I_i]/T)\}. \tag{2}$$

Similarly

$$p(s_i^{n+1}=\pm 1|s_i^n=-1)=1/\{1+\exp(\mp 2[h_i^n-I_i]/T)\}. \tag{3}$$

Here T denotes temperature of the system in energy unit with the Boltzmann constant set to 1. If there is no SI, i.e., $I_i=0$ $(i=1,2,\ldots,N)$ our system satisfies the detailed balance relation and the system relaxes to the canonical equilibrium distribution $p_{\text{eq}}(s_1,\ldots,s_N)\propto\exp[-H(s_1,\ldots,s_N)/T]$ with $H(s_1,\ldots,s_N)=-(1/2)\Sigma_{i,j(\neq i)}J_{i,j}s_i s_j$ [2,5].

Introducing the two-dimensional probability vector $\boldsymbol{p}_i^n=[p(s_i^n=-1),p(s_i^n=1)]^{\mathcal{T}}$ with $p(s_i^n=-1)=1-p(s_i^n=1)$ denoting the probability that $s_i^n=-1$ and the suffix $\mathcal{T}$ transpose, one can express Glauber dynamics (2) and (3) more concisely as

$$\boldsymbol{p}_i^{n+1}=A\boldsymbol{p}_i^n, \tag{4}$$

where the 2×2 matrix A has the elements

$$A_{1,1}=[1+\exp(2M_i^n)]^{-1}\equiv\alpha_i,$$

$$A_{1,2}=[1+\exp(2P_i^n)]^{-1}\equiv\beta_i,$$

$$A_{2,1}=[1+\exp(-2M_i^n)]^{-1}=1-\alpha_i,$$

$$A_{2,2}=[1+\exp(-2P_i^n)]^{-1}=1-\beta_i, \tag{5}$$

with $P_i^n\equiv(h_i^n+I_i)/T$ and $M_i^n\equiv(h_i^n-I_i)/T$. For later convenience [see Eq. (18)] α_i and β_i are introduced in Eq. (5) with their time (i.e., n) dependence omitted.

Since the magnetization of the spin i is defined by

$$m_i^n=\langle s_i^n\rangle\equiv p(s_i^n=1)-p(s_i^n=-1), \tag{6}$$

we have from Eqs. (4) and (5) that

$$m_i^{n+1}=p(s_i^n=1)\tanh P_i^n+p(s_i^n=-1)\tanh M_i^n$$
$$=(m_i^n/2)[\tanh P_i^n-\tanh M_i^n]+(1/2)[\tanh P_i^n+\tanh M_i^n]. \tag{7}$$

Now let us consider a physical role of the SI in Eq. (1), with $I_i (i=1,\ldots,N)$ assumed hereafter to be positive. If $s_i^n=1$, the spin feels the field $h_i^n+I_i$ and the self-part I_i tends to keep the spin in the present state $s=1$. Similarly if $s_i^n=-1$, the self-part $-I_i$ of the field on the spin $h_i^n-I_i$ is against flipping of the spin to $s=1$. Thus $I_i(>0)$ may be considered to represent a kind of activation energy or friction and its effect is flip inhibiting.

To see this point more clearly, we tentatively consider a noninteracting system ($J_{i,j}=0$). Then the matrix A in Eq. (5) is reduced to $\tilde{A}$ with the elements $\tilde{A}_{1,1}=\tilde{A}_{2,2}=1/[1+\exp(-2I_i/T)]$ and $\tilde{A}_{1,2}=\tilde{A}_{2,1}=1/[1+\exp(2I_i/T)]$. $\tilde{A}$ has an eigenvalue $\lambda_1=1$ with an eigenvecor $(1/\sqrt{2},1/\sqrt{2})^{\mathcal{T}}$, representing an equilibrium state, and another eigenvalue $\tanh(I_i/T)$ with an eigenvector $(1/\sqrt{2},-1/\sqrt{2})^{\mathcal{T}}$, representing a relaxation mode. The relaxation time τ is defined to be $1/|\ln\lambda_2|$ [8], which reduces to $\tau\simeq\exp(2I_i/T)$ when $2I_i/T\gg 1$. Thus we see that $2I_i$ may be interpreted as an activation energy for the spin s_i to flip.

Since we are mainly interested in effects of SI, we hereafter consider the interaction part of the field f_i^n, Eq. (1), within the mean field model, $J_{i,j}=J/N$, leading to

$$h_i^n = JS^n, \tag{8}$$

where

$$S^n = \sum_j s_j^n/N. \tag{9}$$

Finally we introduce (quenched) disorder in $I_i(i=1,2,\ldots,N)$ by assuming that the strength I_i of SI is distributed according to the exponential law

$$p(I_i) = (1/T_0)\exp(-I_i/T_0), \tag{10}$$

where T_0 has the dimension of temperature. In passing it is noted that a distribution similar to Eq. (10) is considered before to discuss hopping conduction of a classical particle by Bernasconi *et al.* [9].

III. PHASE DIAGRAM

Since the interaction parameter $J(>0)$ is specified to be, say 1, the thermodynamic state (phase diagram) depends on two parameters T and T_0 and it can be discussed based on Eq. (7) as follows.

When n in Eq. (7) is very large, we may put $m_i^n=m_i^{n+1}=m_i$ and we have for $N\to\infty$

$$m_i = (\tanh P_i + \tanh M_i)/(2+\tanh M_i - \tanh P_i) \equiv g(m,I_i|T), \tag{11}$$

where from Eqs. (7) and (8) $P_i=(m+I_i)/T$ and $M_i=(m-I_i)/T$ with $m\equiv\sum m_i/N$. Since I_i is distributed according to Eq. (10), we have a self-consistent equation to determine the magnetization m as a function of T and T_0:

$$m = \int dI' p(I') g(m,I'|T). \tag{12}$$

In Fig. 1 we plot m, calculated from Eq. (12), as a function of T for (a) $2T_0=1.5$ and (b) $2T_0=1.0$. The critical temperature is seen to be $T_c\simeq 1.4$ (a) and $T_c\simeq 1.3$ (b). When $I_i=0$ $(i=1,2,\ldots,N)$, we know that $T_c=1.0$ for $J=1$. To see effects of the self-interaction I on the shift of T_c from $T_c=1.0$, we tentatively consider a system with no disorder in $\{I_i\}$, that is $I_i=I(i=1,2,\ldots,N)$. In this case we have, by putting $p(I')=\delta(I'-I)$ in Eq. (12), the self-consistent equation

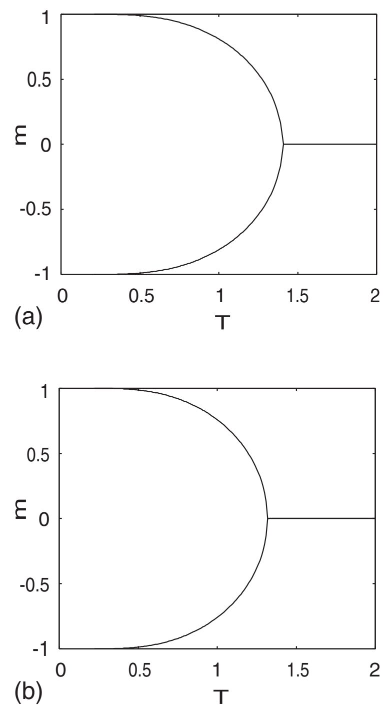

FIG. 1. Magnetization m as a function of T for (a) $2T_0=1.5$ and for (b) $2T_0=1.0$ with disorder given by Eq. (10).

$$m = g(m,I|T). \tag{13}$$

The critical temperature T_c as a function of I is given by the equation

$$dg\,(m,I|T_c)/dm|_{m=0} = 1, \tag{14}$$

leading to

$$T_c = [\cosh^2(I/T_c)(1-\tanh(I/T_c))]^{-1}. \tag{15}$$

We observe from Eq. (15) that $T_c=1$ for $I=0$ and it goes to 1 ± 1 as I goes to $\pm\infty$. The activation energy $I(>0)$ inhibits flipping and this raises T_c, which is consistent with the shift of T_c shown in Fig. 1. In passing it is noted that negative I enhances flipping and we must cool down the system in order to have spontaneous magnetization, thus $T_c\to 0$ as $I\to-\infty$.

From the above we see that the system (1) with Eqs. (8) and (10) has a relatively simple phase diagram (Fig. 1) for $N\to\infty$.

IV. DYNAMICAL PROPERTIES: TCF

We now turn our attention to dynamical properties of the system and consider the equilibrium TCF,

$$\phi(n) = \sum_i \langle s_i^n s_i^0 \rangle / N,$$

$$= \sum_i \sum_{s_i^0 = \pm 1} \langle s_i^n \rangle_{s_i^0} s_i^0 p_{\rm eq}(s_i^0)/N, \tag{16}$$

where the average $\langle \cdots \rangle$ is over the equilibrium distribution function $\Pi_{i=1,N} p_{\rm eq}(s_i^0)$, where $p_{\rm eq}(s_i^0=1)=(1+m_i)/2$ and $p_{\rm eq}(s_i^0=-1)=(1-m_i)/2$, with m_i given as a solution to Eqs. (11) and (12). $\langle s_i^n \rangle_{s_i^0}$ on the right-hand side (RHS) of Eq. (16) denotes the average of s_i^n when initially it starts from the state s_i^0, which can be expressed for, e.g. $s_i^0=1$, as

$$\langle s_i^n \rangle_{s_i^0=1} = [A^n \boldsymbol{p}_0]_2 - [A^n \boldsymbol{p}_0]_1. \tag{17}$$

Here $\boldsymbol{p}_0=(0,1)^{\mathcal{T}}$ and $[\boldsymbol{p}]_2$ denote the second element of the probability vector $\boldsymbol{p}$ [see Eq. (6)].

We first note that the (equilibrium) magnetization m and thus the matrix A, Eq. (5), are time (n) independent as it should be since we are dealing with the equilibrium TCF in the thermodynamic limit. With use of the eigenvalue $\lambda_1=1$ and $\lambda_2=\alpha_i-\beta_i$ with $|\lambda_1\rangle=[\beta_i/(1-\alpha_i),1]^{\mathcal{T}}$, $\langle\lambda_1|=(1-\alpha_i)/(1+\beta_i-\alpha_i)(1,1)$, and $|\lambda_2\rangle=-\beta_i/(1+\beta_i-\alpha_i)(1,-1)^{\mathcal{T}}$, $\langle\lambda_2|=[-(1-\alpha_i)/\beta_i,1]$, we have

$$A^n = |\lambda_1\rangle\langle\lambda_1| + \lambda_2^n |\lambda_2\rangle\langle\lambda_2|, \tag{18}$$

or more explicitly we obtain from Eqs. (16)–(18)

$$\bar{\phi}(n) \equiv \phi(n) - \int d\, {\rm I} m^2(I) p(I)$$

$$= \int dI[\alpha(I)-\beta(I)]^n \times [1-m^2(I)]p(I), \tag{19}$$

where we have replaced $\Sigma_i m_i^2/N$ and $\Sigma_i(\alpha_i-\beta_i)^n(1-m_i^2)/N$ by their integral forms in the limit $N\to\infty$.

When $T>T_c$ we have from Eqs. (11)–(13) that $m_i=0 (i=1,2,\ldots,N), m=0$, and

$$p_{\rm eq}(s_i^0=1) = p_{\rm eq}(s_i^0=-1) = 1/2 \quad (i=1,\ldots,N). \tag{20}$$

In this case Eq. (19) is reduced with Eq. (18) to

$$\phi(n) = \int dI p(I) \tanh^n(I/T). \tag{21}$$

Exact calculation can be easily performed for time integral

$$A \equiv \sum_n \phi(n) = \int dI p(I)[1-\tanh(I/T)]^{-1}, \tag{22}$$

leading to ($T>T_c$)

$$A = [1 + T/(T-2T_0)]/2 \quad (T>2T_0),$$

$$A = \infty \quad (T<2T_0). \tag{23}$$

The correlation function (21) can be estimated by using a saddle point method to obtain for $n \gg 1$

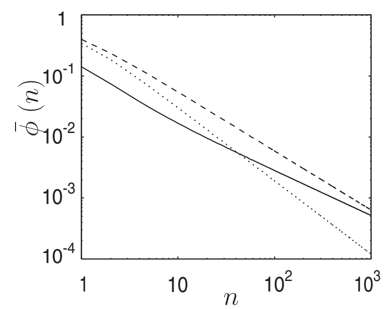

FIG. 2. $\bar{\phi}(n)$, Eq. (19), as a function of time n for $T=1.8$ (dotted curve), $T=1.45$ (dashed curve), and $T=1.1$ (full curve) in the case $2T_0=1.5$.

$$\phi(n) = (1/T_0)\exp(-T/2T_0)[4nT_0/T]^{-T/2T_0} \propto n^{-T/2T_0}, \tag{24}$$

with the saddle point $I_s(n)$ given by

$$I_s(n) = (T/2)\ln[4nT_0/T]. \tag{25}$$

Equation (23) shows that $\phi(n)$ goes to zero faster than $1/n$ for $T>2T_0$ but it relaxes to zero slower than $1/n$ for $T<2T_0$, consistent with the exact result [Eq. (23)]. It is noted that we obtain, following similar asymptotic analysis, the same algebraic relaxation (24) for $T<T_c$.

Numerical calculations of $\phi(n)$ based on Eq. (19) are straightfoward once one solves the self-consistent Eq. (12) with Eq. (11). In Fig. 2 we depict the relaxation part of $\phi(n)$, i.e., the RHS of Eq. (19), for $T=1.8$ ($m=0$), $T=1.45$ ($m=0$), and $T=1.1$ ($m=\pm 0.714$) for the case $2T_0=1.5$. Asymptotic algebraic relaxation is read from Fig. 2 as $n^{-1.2}$, $n^{-0.967}$, and $n^{-0.735}$, which should be compared with $n^{-1.2}$, $n^{-0.966}$, and $n^{-0.733}$ from Eq. (24).

Now we turn to consider effects of finite system size on the TCF $\phi(n)$. To consider this problem, we briefly review statistics of the maximum value among N random variables, which are distributed identically and independently [10]. That is, we have N random variable, $I_i (i=1,\ldots,N)$ distributed independently according to Eq. (10). Then the maximum I_M among the N random variables is distributed, following the probability distribution [10]

$$P(I_M) = N[p_<(I_M)]^{N-1}\exp(-I_M/T_0)/T_0, \tag{26}$$

where

$$p_<(X) \equiv \int_0^X dI p(I) = [1-\exp(-X/T_0)]. \tag{27}$$

Thus we can estimate the most probable value, which we call $I_{\rm MP}$, for I_M from $dP(X)/dX|_{I_{\rm MP}}=0$ to obtain

$$I_{\rm MP}(N) = T_0 \ln N. \tag{28}$$

The upper limit, i.e., ∞, in the I integral in Eqs. (19) and (21) must be replaced by $I_{\rm MP}=T_0 \ln N$ for a finite system with size N. In Fig. 3 we plot $\phi(n)$ thus obtained for $N=10^2$ and $N=10^3$ together with $N=\infty$. For a system with smaller N the deviation from $\phi(n)$ for $N=\infty$ occurs earlier as

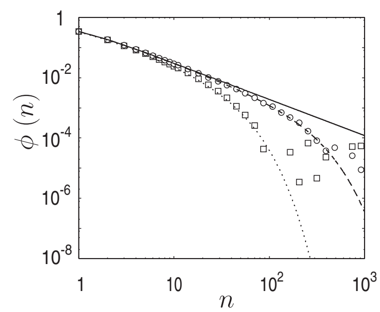

FIG. 3. $\phi(n)$ for $2T_0=1.5$ and $T=1.8$ for $N=10^2$ (dotted curve) and $N=10^3$ (dashed curve). Monte Carlo results are shown by squares ($N=10^2$) and circles ($n=10^3$).

expected. If we fix time n in Eq. (19) or (21), the saddle point, given by Eq. (25), must be smaller than I_{MP} in order for the saddle point method to be applicable. From this we can say that the asymptotic behavior (24) is valid only for $n<(T/4T_0)N^{2T_0/T}$.

To study effects of finite system size numerically, we performed Monte Carlo simulations, in which one updates each probability vector $\boldsymbol{p}_i^n$ according to Eq. (4), with $\boldsymbol{p}_i^0$ either taking the value $(1,0)^{\mathcal{T}}$ or $(0,1)^{\mathcal{T}}$ with the probability $p_{eq}(s_i=-1)$ and $p_{eq}(s_i=1)$, respectively. Magnetization m^n is updated by $m^n=\Sigma m_i^n/N$ with $m_i^n=[\boldsymbol{p}_i^n]_2-[\boldsymbol{p}_i^n]_1$, starting from m^0 chosen to be the solution of Eq. (12) [see Eq. (16)]. In Fig. 3 simulation results are also shown, which were obtained as an ensemble average over 10^5 (10^6) samples for the system with $N=10^3$ (10^2) spins. We observe good correspondence between theory and experiment except for the time (n) region where $\phi(n)$ becomes of the order 10^{-4} and the fluctuation in $\phi(n)$ becomes large.

V. SOME REMARKS

In this paper we studied effects of SI on static and dynamic properties in a simple spin system. Various properties, related to an equilibrium phase diagram and the TCF, of the system can be qualitatively understood once we interpret the SI as an activation energy.

In this connection we briefly touch upon two models, which may have some overlapping with our model, Eqs. (1)–(3). To study dynamics of price in stock markets, Kaizoji *et al.* [6] proposed a spin model in which the field f_i^n takes the form

$$f_i^n = \sum_{j\neq i} J_{i,j}s_j^n + bs_i^n|m^n|, \tag{29}$$

with b a constant. Comparing Eq. (29) with Eq. (1) we notice that SI strength I_i in Eq. (1) is replaced by $b|m|$, whose effects is discussed in connection with traders' tendency of switching between majority and minority groups.

A simple threshold system with feedback [8] gives another example of SI. Here the output at time n, y^n, in response to the input signal s^n is expressed in terms of the Heaviside function $\Theta(x)$ as

$$y^n = \Theta(s^n + Fy^{n-1} + \xi^n - \theta), \tag{30}$$

where $\Theta(x)=1$ for $x>0$ and $\Theta(x)=0$ for $x<0$. Here F and θ denote the feedback strength and a threshold value, respectively, and the noise ξ^n is assumed to be Gaussian white with the variance σ. If we introduce temperature T of the system by $T\equiv 0.4\sigma\sqrt{2}$, we notice that the transition probability Eq. (23) in Ref. [8] has similar form as Eq. (2) with F playing the role of SI. Thus feedback F in a simple information processing device turns out to cause friction to flipping in the output and this point, that is, how the system output y^n follows the time-dependent input signal s^n, is discussed in [8]. When F becomes large, the timescale of variation of y^n becomes slow with enhanced retardation and hysteresis in y^n.

From these examples it is seen that SI in spin systems may give rise to interesting phenomena and we hope that our work here may stimulate further studies in neural networks and other simple interacting threshold systems [11].

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